The Counter to Equilibrium

Task 2: Equal Distribution

Submitted by: Mohammad S Anwar

You are given a list of integers greater than or equal to zero, $@list$.

Write a script to distribute the number so that each members are same. If you succeed then print the total moves otherwise print -1.

Please follow the rules (as suggested by Niels 'PerlBoy' van Dijke)

- You can only move a value of 1 per move
- You are only allowed to move a value of 1 to a direct neighbor/adjacent cell

Example 1

Input: $@list = (1, 0, 5)$
Output: 4

Move #1: $(1, 1, 4)$
(2nd cell gets 1 from the 3rd cell)
Move #2: $(1, 2, 3)$
(2nd cell gets 1 from the 3rd cell)
Move #3: $(2, 1, 3)$
(1st cell gets 1 from the 2nd cell)
Move #4: $(2, 2, 2)$
(2nd cell gets 1 from the 3rd cell)

Example 2

Input: $@list = (0, 2, 0)$
Output: -1

It is not possible to make each same.

Example 3

Input: $@list = (0, 3, 0)$
Output: 2

Move #1: $(1, 2, 0)$
(1st cell gets 1 from the 2nd cell)
Move #2: $(1, 1, 1)$
(3rd cell gets 1 from the 2nd cell)

Definitions

A *move*, i.e. the transfer of a unit from one element to one of its immediate neighbors will be called *shift left* or *shift right* in the following.

Given a distribution $(a_1, \ldots, a_n)$, we define the *cumulative distribution*

$$(s_1, \ldots, s_n) := (a_1, a_1 + a_2, \ldots, a_1 + \ldots + a_n) = \left(\sum_{i=1}^{k} a_i \right)_{k=1}^{n}$$

Properties

Moves

Consider an element $a_i > 0$.

We can perform a *shift left* from position i to $i - 1$ if $i > 1$:

$$a'_k = \begin{cases} a_k + 1 & \text{if } k = i - 1 \\ a_k - 1 & \text{if } k = i \\ a_k & \text{otherwise} \end{cases}$$

In terms of s:

$$s'_k = \begin{cases} s_k + 1 & \text{if } k = i - 1 \\ s_k & \text{otherwise} \end{cases}$$

We can perform a *shift right* from positon i to $i + 1$ if $i < n$:

$$a'_k = \begin{cases} a_k - 1 & \text{if } k = i \\ a_k + 1 & \text{if } k = i + 1 \\ a_k & \text{otherwise} \end{cases}$$

In terms of s:

$$s'_k = \begin{cases} s_k - 1 & \text{if } k = i \\ s_k & \text{otherwise} \end{cases}$$

Effects

Now we know exactly how a single move affects the cumulative distribution:

- A *shift left* increments the cumulative distribution at the target position by one.
- A *shift right* decrements the cumulative distribution at the source position by one.
- All other values of the cumulative distribution remain unchanged.

Disequilibrium

There exists an equilibrium distribution if and only if

$$s_n \bmod n = 0$$

Let's assume there is an equilibrium distribution for the given numbers. Then there is a *cumulative equilibrium distribution*

$$(e_1, \ldots, e_n) := (s_n/n, 2s_n/n, \ldots, s_n) = (ks_n/n)_{k=1}^n$$

Note that $e_n = s_n$ and $e_i < e_{i+1}$ for $i = 1, \ldots, n-1$.

Considering the cumulative distribution:

1. Suppose there is an i with $s_i > e_i$. Then $i < n$ because $s_n = e_n$. If $a_i > 0$ then a *shift right* from position i to $i+1$ decrements s_i by one. Otherwise (if $a_i = 0$), $s_{i-1} = s_i > e_i > e_{i-1}$ and we may repeat the consideration at $i-1$. At least one of $a_1, \ldots, a_i$ must be non-zero because otherwise $s_i > e_i$ cannot hold.

   Decrementing:
   If there is an i with $s_i > e_i$, then there is a j with $1 \leq j \leq i$, $a_j > 0$ and $s_i = s_j > e_j$. With a *shift right* from position j to $j+1$, s_j can be decremented by one.

   Example:

   $$\begin{aligned} a &= (2, 7, 0, 0, 1) \\ s &= (2, 9, 9, 9, 10) \\ e &= (2, 4, 6, 8, 10) \end{aligned}$$

   Select $i = 4$: $s_4 = 9 > 8$ but $a_4 = 0$. Looking left, we arrive at $j = 2$ having $s_2 = 9 > 4$ and $a_2 > 0$ where we can perform a *shift right* from position 2 to 3 resulting in

   $$\begin{aligned} a' &= (2, 6, 1, 0, 1) \\ s' &= (2, 8, 9, 9, 10) \end{aligned}$$

2. Suppose there is an i with $s_i < e_i$.

   Then $i < n$ because $s_n = e_n$. If $a_{i+1} > 0$ then a shift left from position $i+1$ to i increments s_i by one. Otherwise (if $a_{i+1} = 0$), $s_i = s_{i+1} < e_i < e_{i+1}$ and we may repeat the consideration at $i+1$. At least one of $a_{i+1}, \ldots, a_n$ must be non-zero because otherwise $s_i < e_i$ cannot hold.

   Incrementing:
   If there is an i with $s_i < e_i$, then there is a j with $i \leq j < n$, $a_{j+1} > 0$ and $s_i = s_j < e_j$. With a *shift left* from position $j+1$ to j, s_j can be incremented by one.

Example:

$$\begin{aligned} a &= (1,0,0,7,2) \\ s &= (1,1,1,8,10) \\ e &= (2,4,6,8,10) \end{aligned}$$

Select $i = 1$: $s_1 = 1 < 2$ but $a_2 = 0$ Looking right, we arrive at $j = 3$ having $s_3 = 1 < 6$ and $a_4 > 0$ where we can perform a *shift left* from position 4 to 3 resulting in

$$\begin{aligned} a' &= (1,0,1,6,2) \\ s' &= (1,1,2,8,10) \end{aligned}$$

Existence

From the previous section we already know that a single move increments or decrements exactly one element of the cumulative distribution by one.

Now we have shown that in a distribution deviating from the cumulative equilibrium distribution, there always exists a legal move that reduces the overall deviation.

Solution

We have shown that for every deviation from the equilibrium distribution there is a move that reduces the difference between the cumulative distribution and the cumulative equilibrium distribution by one. The number of moves to achieve the equilibrium distribution is therefore the sum of the absolute differences between the cumulative distribution and the cumulative equilibrium distribution:

$$m = \sum_{i=1}^{n} |s_i - e_i|$$

This is a nice result for mathematicians: It gives the required number of moves without actually providing them. (However, from the above discussion it becomes clear that any legal move reducing the overall deviation may be chosen at any time.)